Localization

Recall: A <u>local ring</u> is a ring with exactly one maximal ideal.

Often it is easier to work w/ local rings – the units are exactly the elements not in the maximal ideal.
We can sometimes reduce to the local case by "localizing" a ring, essentially by adding inverses for each elt outside of a given ideal.

Question: For which elements can we add inverses?

If we add f^{-1} and g^{-1} then we're also adding $(fg)^{-1}$, so the set of elts U whose inverses we add must be <u>multiplicatively closed</u> (i.e. products of elts in U are in U including the "empty product" 1).

Ex:

1.) If $t \in R$, $t \neq 0$, $\{1, t, t^2, \ldots\}$ is multiplicatively closed.

2.) $P \subsetneq R$ an ideal. $R - P$ is mult. closed iff P is prime.

3.) $R - \{0\}$ is mult. closed iff R is an integral domain.

Def: Let M be an R-module, $U \subseteq R$ multiplicatively closed. The localization of M at U, $M[U^{-1}]$ or $U^{-1}M$, is the set of the equivance classes of pairs $m \in M$, $u \in U$ (written $\frac{m}{u}$) w/ equivalence relation

$$\frac{m}{u} \sim \frac{m'}{u'} \iff \exists\, v \in U \text{ s.t. } vu'm = vum' \text{ in } M.$$

$M[U^{-1}]$ is an R-module by defining

$$\frac{m}{u} + \frac{m'}{u'} = \frac{u'm + um'}{uu'} \quad \text{and} \quad r\left(\frac{m}{u}\right) = \frac{rm}{u}.$$

In fact, $M[U^{-1}]$ is an $R[U^{-1}]$-module in the obvious way:

$$\left(\frac{r}{u}\right)\left(\frac{m}{u'}\right) = \frac{rm}{uu'}$$

(If $U \subseteq R$ is an arbitrary set, we can take its mult. closure $\bar{U}$, and define $M[U^{-1}] = M[\bar{U}^{-1}]$)

Note: What happens if $u \in U$, $m \in M$ s.t. $um = 0$? Then $\frac{m}{1} = 0$.

In fact, the converse holds: If $\frac{m}{1} = 0$, $\exists\, v \in U$ s.t. $vm = 0$.

Ex: 1.) For an integral domain, $R[(R-\{0\})^{-1}]$ is the field of fractions, or quotient ring of R, denoted $K(R)$.

2.) More generally, if P is a prime ideal, write $R_p := R[(R-P)^{-1}]$. This is a local ring since the units are exactly the elts not in P.

If M is an R-module, then $M_p := M[(R-P)^{-1}]$ is an R_p-module.

Localization as a functor

If $\varphi: M \to N$ is a map of R-modules, and $U \subseteq R$ multiplicatively closed, there is a natural map

$$\varphi[U^{-1}]: M[U^{-1}] \to N[U^{-1}] \quad \text{s.t.} \quad \frac{m}{u} \mapsto \frac{\varphi(m)}{u}$$

of $R[U^{-1}]$-modules.

Check: $L \xrightarrow{\psi} M \xrightarrow{\varphi} N \implies (\varphi \circ \psi)[U^{-1}] = \varphi[U^{-1}] \circ \psi[U^{-1}]$, so localization is a functor from R-modules to $R[U^{-1}]$-modules.

Universal property

Suppose $\varphi: R \to S$ is a ring homomorphism, $U \subseteq R$ mult. closed.

Then as long as U gets sent to units in S, we can uniquely extend φ to

$\varphi': R[U^{-1}] \to S[\varphi(U)^{-1}]$ by

$\frac{a}{b} \mapsto \varphi(a)\varphi(b)^{-1}$.

This is the <u>universal property</u> of localization. i.e.

if

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr] & S \\ & R[U^{-1}] \arrow[u, dashed, "\varphi'"'] \end{tikzcd}$$

sends U to units, $\exists$ unique φ' st commutes.

Ideals of $R[U^{-1}]$

let $\varphi : R \to R[U^{-1}]$ be the natural map.

If $I \subseteq R[U^{-1}]$, then $\frac{r}{u} \in I \Rightarrow r \in I$, so $r \in \varphi^{-1}(I)$.

Thus, all numerators are in $\varphi^{-1}(I) \Rightarrow I$ is the ideal generated by $\varphi^{-1}(I)$.

$\Longrightarrow I \mapsto \varphi^{-1}(I)$ is an injection.

Which ideals of R are the preimages of ideals in $R[U^{-1}]$?

$J \subseteq R$ is the preimage of an ideal iff $J = \varphi^{-1}(JR[U^{-1}])$.

This won't be the case iff $\exists$ $b \in J$, $u \in U$, s.t.

$\frac{a}{1} = \frac{b}{u}$, and $a \notin J$.

i.e. $u'(ua - b) = 0$, some $u' \in U$.

In other words, J is such a preimage iff there is no

$u \in U$, $a \notin J$ s.t. $au \in J$. In particular, since preimages of primes are prime...

Prop: The correspondence $I \mapsto \varphi(I)$ is a bijection on prime ideals avoiding U.

Ex: If $P \subseteq R$ is prime, then the prime ideals of R_P are in one-to-one correspondence w/ primes of R contained in P.

Rmk: Recall that the primes of R/I corr. to the primes in R that contain I.

Cor: If R is Noetherian, so is $R[U^{-1}]$.